Numerical two-dimensional flexible channel model fixed at both ends for flow-induced instability analysis

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Abstract: The proposed study arises from a fluid-structure interaction (FSI) paradox of whether a fluid-conveying flexible channel fixed at both ends can experience flutter instability. Following the initial identification of this paradox in the 1970s, there have been findings that support the occurrence of flutter, both experimentally and analytically. Substantial evidence has also accumulated refuting the possibility of flutter occurring. Presently, this issue has yet to be satisfactorily resolved. In this paper a model of a two-dimensional channel with a flexible segment pinned at both ends is developed. The governing equations for the fluid, solid and their interaction are detailed as well as the specifics of their numerical solution. Preliminary results demonstrate the validity of the numerical model. This paper establishes the framework for a more detailed investigation of the FSI system.

Keywords: computational fluid dynamics, finite element analysis, fluid-structure interaction.

1 Introduction

The investigation regarding possible occurrence of flutter for a fluid-conveying flexible pipe with fixed ends has great significance in understanding a fundamental phenomenon of Nature. It has many biomechanical applications as flexible conduits are universal in the human body. Examples of these are the arterial, venous, lymphatic, pulmonary airway and urinary systems [1].

There has been a long standing paradox on this problem since the 1970s [2]. For a flexible pipe with fixed ends conveying a fluid it has been argued through theoretical analysis that energy conservation does not permit flutter to occur [3, 4, 5]. On the other hand, it has been demonstrated experimentally that flutter behaviour can exist [4, 6]. Theories have also been developed that predict flutter occurrence [7-9]. Thus a paradox arises from the seemingly plausible arguments of both explanations. Although better understanding has emerged through past investigations, it has still to be proven with adequate confidence which behaviour is true.

The characteristic phenomena can be reproduced in laboratory experiments using the Starling Resistor [6]. A simpler precursor model was first introduced by Pedley [10] and is shown in Figure 1. It consists of a two-dimensional (2-d) channel with one segment of the wall replaced by a membrane under longitudinal tension. There are practical difficulties in producing the 2-d flow experimentally. However it still has considerable theoretical advantages as it avoids the complications of fully 3-d flows found in the Starling Resistor while exhibiting flow limitation and self-excited oscillations [11].

The study of fluid-structure interaction (FSI) problems using numerical methods has become increasingly popular due to advances in computational power. The Object-Oriented Multi-Physics Finite-Element Library (oomph-lib), an open-source project, suits the objectives of this study [12] and is well-established as a research tool for the study of fluid-structure interactions.

This objective of this paper is to develop a versatile model through which flutter in the Pedley [10] FSI system can be studied, as a first step towards the aim of modelling a planar tube of which both walls are
flexible. The equations are solved numerically with oomph-lib. In Section 2 we introduce the governing equations and their numerical implementation including the characteristic scales used for non-dimensionalisation. The input parameters are summarised in Section 3. A preliminary investigation is presented in Section 4 containing benchmarks of the pressure and velocity fields. We conclude with Section 5 and outline the next stage of this investigation.

Figure 1: Schematic of the 2-d model: viscous flow through a channel with part of one wall replaced by an elastic membrane, adapted from [11]

2 Governing Equations and Boundary Conditions

The model created is based on that studied by Pedley [10]. For the present investigation, Figure 2 shows the main geometrical parameters of the model. Variables identified with asterisks are dimensional and those without asterisks are non-dimensional. Fluid flow is driven by a prescribed Poiseuille velocity profile at the inlet of the 2-d channel of width $H^*$ and total length $L^\text{total}$. The total length is the summation of the upstream length $L^\text{up}$, collapsible section length $L^\text{collapsible}$, and downstream length $L^\text{down}$. The upstream and downstream sections are rigid, and the central section is an elastic membrane. The wall is loaded by an external pressure $p^\text{ext}$ and the traction that the fluid exerts on it.

The governing equations for the fluid and flexible wall components are non-dimensionalised to better condition the numerical problem in oomph-lib. Table 1 lists the quantities used for the non-dimensionalisation.

Figure 2: Geometry of the present 2-d model of a one-sided flexible channel fixed at both ends
Table 1: Scaling Quantities for Non-dimensionalisation.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Scaling Quantity</th>
<th>Mathematical Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>Channel width</td>
<td>( H^* )</td>
</tr>
<tr>
<td>Velocity</td>
<td>Average velocity through undeformed channel</td>
<td>( U_{\text{mean}}^* = \frac{P^* H^{<em>2}}{(12 \mu^</em> L_{\text{Total}}^*)} )</td>
</tr>
<tr>
<td>Time</td>
<td>Division of channel width with velocity</td>
<td>( T^* = \frac{H^<em>}{U_{\text{mean}}^</em>} )</td>
</tr>
<tr>
<td>Pressure</td>
<td>Viscous scale</td>
<td>( \mu^* \frac{U_{\text{mean}}^<em>}{H^</em>} )</td>
</tr>
<tr>
<td>Stresses and Applied traction</td>
<td>Effective Young’s modulus</td>
<td>( E_{\text{eff}}^* = \frac{E^*}{(1 - \nu^2)} )</td>
</tr>
</tbody>
</table>

The non-dimensional parameter

\[
Q = \frac{\mu^* U^*}{E_{\text{eff}}^* H^*}
\]

is the ratio of the fluid pressure scale, \( \mu^* U^* / H^* \), to a measure of membrane-material stiffness, \( E_{\text{eff}}^* \), and used to characterize the fluid-structure interaction in the formulation of the coupled problem.

2.1 Fluid

The fluid is governed by the incompressible Navier-Stokes equation and continuity,

\[
\rho^* \frac{d\mathbf{V}^*}{dt} = -\nabla^* p^* + \mu^* \left( \nabla^* \right)^2 \mathbf{V}^* \quad \text{and} \quad \nabla^* \mathbf{V}^* = 0 ,
\]

with density \( \rho^* \) and dynamic viscosity \( \mu^* \); and variables pressure \( p^* \) and velocity [14]

\[
\mathbf{V}^* = u_1^* \mathbf{e}_1 + u_2^* \mathbf{e}_2 ,
\]

where \( u \) is the velocity component and \( e \) is the unit vector; subscripts 1 and 2 respectively denote the horizontal and vertical directions as shown in the axes of Figure 1. The fluid is assumed to be Newtonian and incompressible.

Non-dimensionalisation using the entries of Table 1 gives the Navier-Stokes equation in the following form [13].

\[
\text{Re} \left( \text{St} \frac{\partial \mathbf{u}^*}{\partial t} + u_j^* \frac{\partial \mathbf{u}^*}{\partial x_j} \right) = -\frac{\partial p^*}{\partial x_i} + \frac{\partial u_i^*}{\partial x_j} \left( \frac{\partial u_j^*}{\partial x_j} + \frac{\partial u_i^*}{\partial x_j} \right)
\]

and the continuity equation

\[
\frac{\partial u_i^*}{\partial x_i} = 0 ,
\]

with Reynolds Number \( \text{Re} \)

\[
\text{Re} = \frac{\rho^* U_{\text{mean}}^* H^*}{\mu^*}
\]

and Strouhal number set to unity as defined by
\[
St = \frac{H^*}{U_{mean}^*} = 1. \quad (7)
\]

The fluid flow is subject to the following initial and boundary conditions:

- Inflow is prescribed to be a plane Poiseuille velocity profile
  \[V(x_1 = 0, x_2, t = 0) = 6x_2(1 - x_2)\xi_1 + 0\xi_2; \quad (8)\]

- No slip on all channel walls, for rigid walls:
  \[V = 0, \quad (9)\]

- and for flexible walls:
  \[V = St \frac{\partial R_w}{\partial t} \quad (10)\]

where \(R_w\) is the flexible wall displaced position.

The finite-element model used to solve the Navier-Stokes equations discretises the fluid domain with 2-d Taylor-Hood elements. Nodal positions are updated in response to the changes in the flexible-wall position when it deforms.

### 2.2 Flexible Channel Wall

The flexible-wall dynamics are modeled using Kirchoff-Love beam theory with incrementally linear constitutive equations. The computational model in oomph-lib discretises the flexible wall using one-dimensional, isoparametric, two-node Hermite beam elements.

The beam’s undeformed shape is parameterised by a non-dimensional Lagrangian coordinate \(\xi\) and the non-dimensional position vector to a material particle on the beam’s centerline in the undeformed configuration is given by \(r_w(\xi)\). The unit normal to the beam’s undeformed centerline is denoted \(n\). The applied traction that deforms the beam causes its material particle to be displaced to the new position \(R_w(\xi)\), and the unit normal to the beam’s centerline is \(N\) [13].

The non-dimensional form of the principle of virtual displacements that governs the beam deformation is then given by

\[
\int_0^L \left[ (\gamma) \delta \gamma + \frac{1}{12} h^2 \kappa \delta \kappa - \left( \frac{1}{h} \sqrt{A - \Lambda^2 \frac{\partial^2 R_w}{\partial \xi^2}} \right) \frac{\partial R_w}{\partial \xi} \right] \sqrt{a d \xi} = 0 \quad (11)
\]

where

\[
a = \frac{\partial r_w}{\partial \xi} \cdot \frac{\partial r_w}{\partial \xi} \quad \text{and} \quad A = \frac{\partial R_w}{\partial \xi} \cdot \frac{\partial R_w}{\partial \xi} \quad (12a, b)
\]

represent the squares of the lengths of infinitesimal material line elements in the undeformed and deformed configurations respectively.

We represent the curvature or the beam’s centerline before and after deformation by

\[
b = n \cdot \frac{\partial^2 r_w}{\partial \xi^2} \quad \text{and} \quad B = N \cdot \frac{\partial^2 R_w}{\partial \xi^2}. \quad (13a, b)
\]

The strain and bending “tensors” \(\gamma\) and \(\kappa\) are then given by
\[ \gamma = \frac{1}{2}(A - a) \quad \text{and} \quad \kappa = -(B - b). \quad (14a, b) \]

And the ratio of the natural timescale of the beam’s in-plane extensional oscillations to the fluid-based time scale is defined by

\[ \Lambda = \frac{L^* \sqrt{\rho^* \mu}}{T^* \rho_{eff}}. \quad (15) \]

\( \Lambda^2 \) may also be interpreted as the non-dimensional wall density; thus, setting it is equal to zero corresponds to the case of zero wall inertia.

### 2.3 Coupling and External Pressure

The wall is loaded by an external pressure \( p_{ext}^* \) and the traction that the fluid exerts on it. The components of the load vector \( f^* \) that act on the wall are given by

\[ f_i^* = (p^* - p_{ext}^*) N_i - \mu \left( \frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i} \right) N_j \quad \text{for} \quad i = 1, 2, \quad (16) \]

where \( N_i \) (for \( i = 1, 2 \)) are the components of the outer unit normal on the fluid domain.

The external pressure can later be turned off to create an impulsive disturbance. Following this, the flexible wall may experience several oscillations of decreasing amplitude before settling into a steady position, or have a growth in oscillation amplitude leading to the system becoming unstable. This is the basis of the instability study.

### 3 Property Selection

Parameters for this present study are listed in Table 2. They correspond to the values chosen by Luo et al. [15] which also match the parameters of Davies and Carpenter [16]. These will be used for future comparisons. The resulting non-dimensional parameters used in the oomph-lib model are listed in Table 3.

### 4 Preliminary Validation

The computational model is verified by comparing numerical results with theoretical predictions. Two cases are considered in the following subsections.

#### 4.1 Rigid Channel

The external pressure and FSI are removed. This essentially means the model is a rigid channel with the fluid being driven by a Poiseuille flow at the inlet.
Table 2: Dimensional Parameter Values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_\text{up}^*$</td>
<td>$5 \times 10^{-2}$ m</td>
<td>Upstream length of channel</td>
</tr>
<tr>
<td>$L_\text{collapsible}^*$</td>
<td>$5 \times 10^{-2}$ m</td>
<td>Collapsible length of channel</td>
</tr>
<tr>
<td>$L_\text{down}^*$</td>
<td>$30 \times 10^{-2}$ m</td>
<td>Downstream length of channel</td>
</tr>
<tr>
<td>$H^*$</td>
<td>$1 \times 10^{-2}$ m</td>
<td>Height of channel</td>
</tr>
<tr>
<td>$h^*$</td>
<td>$1.095 \times 10^{-4}$ m</td>
<td>Thickness of flexible wall</td>
</tr>
<tr>
<td>$\rho_f^*$</td>
<td>$1 \times 10^3$ kg.m$^{-3}$</td>
<td>Density of fluid</td>
</tr>
<tr>
<td>$\mu^*$</td>
<td>$1 \times 10^{-3}$ Pa.s$^{-1}$</td>
<td>Dynamic viscosity of fluid</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.5</td>
<td>Poisson’s ratio of solid</td>
</tr>
<tr>
<td>$\rho_s^* \rho_f^*$</td>
<td>0 kg/m$^2$</td>
<td>Mass per unit area of solid</td>
</tr>
<tr>
<td>$B^*$</td>
<td>$7.2 \times 10^{-9}$ N.m</td>
<td>Flexural rigidity of flexible wall</td>
</tr>
<tr>
<td>$U_\text{mean}^*$</td>
<td>0.02 m/s</td>
<td>Inlet mean velocity</td>
</tr>
</tbody>
</table>

Table 3: Non-Dimensional Parameter Values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_\text{up}$</td>
<td>5</td>
<td>Upstream length of channel</td>
</tr>
<tr>
<td>$L_\text{collapsible}$</td>
<td>5</td>
<td>Collapsible length of channel</td>
</tr>
<tr>
<td>$L_\text{down}$</td>
<td>30</td>
<td>Downstream length of channel</td>
</tr>
<tr>
<td>$H$</td>
<td>1</td>
<td>Height of channel</td>
</tr>
<tr>
<td>$h$</td>
<td>$1.095 \times 10^{-2}$</td>
<td>Thickness of flexible wall</td>
</tr>
<tr>
<td>Re</td>
<td>200</td>
<td>Reynolds Number</td>
</tr>
<tr>
<td>St</td>
<td>1</td>
<td>Strouhal Number</td>
</tr>
<tr>
<td>Q</td>
<td>$3.0429 \times 10^8$</td>
<td>FSI parameter</td>
</tr>
<tr>
<td>$\rho_s^* \rho_f^*$</td>
<td>0</td>
<td>Solid-to-fluid density ratio</td>
</tr>
</tbody>
</table>

The oomph-lib model produced a linear pressure drop from inlet to outlet with a non-dimensional inlet pressure of 480, equal to 0.96 Pa when dimensionalised. This value agrees the exact solution of the Navier-Stokes equation for steady laminar flow between parallel planes [17].

Figure 3 shows the velocity profile at the channel mid-length. This demonstrates, as would be expected, that the Poiseuille velocity profile does not change along the channel. The maximum velocity at channel mid-height is confirmed to be 1.5 times the value of the mean velocity and zero at the top and bottom walls as expected [17].

Accordingly, model satisfies the requirements of how the isolated fluid component should act in a rigid-walled system.
4.2 Fixed External Pressure

A constant external pressure of 0.20 Pa is gradually applied on the flexible wall with the Poiseuille flow fully developed throughout the channel. The fluid-structure interaction is disabled which produces a model that tests the behaviour of the fluid when the flexible channel is deformed by the application of an external pressure at the flexible section. All figures presented in this sub-section illustrate the converged steady state of the system.

Figure 4 shows the fluid pressure in the entire channel. The pressure gradient in the rigid-walled section upstream of the flexible wall is seen to be the same as that of the downstream rigid-walled section. At the flexible section a higher rate of pressure drop is observed. This can be seen through the colour band as it decreases from approximately 1000 to 400. Note that Figure 4 also shows the deformation profile of the flexible wall.

Figure 5 shows the centerline pressure variation along the channel. It shows more perceptibly the results of Figure 4, especially the increased pressure-drop rate (negative pressure gradient) in the flexible section. The pressure drop rate is confirmed to be the same in the upstream and downstream sections (at non-dimensional regions along the channel of 0 to 5 and 10 to 40 respectively).

Figure 6 is a plot of the channel centerline horizontal component of velocity against the channel length. Upstream and downstream of the flexible wall, the velocities remain constant at 1.5 as is expected for plane Poiseuille flow in a rigid channel. In the flexible-wall section of the channel there is an increase of horizontal component of velocity to a maximum of approximately 1.64 followed by a decrease to the rigid-wall value of 1.5. This is to be expected on the grounds of mass-conservation through the flexible section that is first convergent and the divergent.
Figure 5: Non-dimensional centerline pressure versus channel length

Figure 6: Centerline horizontal velocity versus distance along channel
Figure 7 shows the variation of the centerline vertical component of fluid velocity along the channel. The upstream and downstream rigid-wall sections show zero vertical velocity as expected for steady laminar flow in a plane channel. The first half of the flexible section has negative vertical velocity and the second half positive vertical velocity. As the cross-section decreases the fluid is directed downward until the middle of the flexible-wall section. The wall cross-section then increases again in the second half of the flexible section which allows the fluid to move upwards. Given that the flexible wall has low curvature (as can be seen in Figure 4), it is reasonable that the change of velocity in this section is also gradual.

Both halves of the flexible section should have produced similar (anti-symmetric) results. However, as can be seen, there is a sharp change at the end of the flexible section. Error is most likely due to the mesh resolution. A more refined mesh in this section should eliminate this problem.

5 Conclusion

This paper has described the development of a computational model of a fluid-conveying channel one side of which has flexible section fixed at its ends. The model has been developed to investigate flow-induced flexible-channel instabilities.

The oomph-lib finite-element toolbox has been shown to be suitable for capturing the behaviours of each of the fluid and structural elements of the system modelled. In particular it enables the investigation of large displacement fluid-structure interactions. Preliminary verification of the model has been shown to produce accurate results as confirmed by theory.

The next step of this study is to introduce a disturbance to the system and determine the conditions that can cause sustained nonlinear oscillations of the flexible wall. Previous studies by Luo et al. [15], and Davies and Carpenter [16] will be used for comparison and validation. Thereafter, a second flexible wall on the opposing side of the channel will be introduced to model a complete flexible channel fixed at both its ends.
6 References


